MATH 207(c)–Analysis IV

1. **(a)** Consider the function defined by

$$f(x) = e^{-ix}\Big(\cos(x) + i\sin(x)\Big)$$

for $x \in \mathbb{R}$. Compute the derivative of f (the function $x \to e^{\lambda x}$ satisfies the same differentiation properties when $\lambda \in \mathbb{C}$ as in the case $\lambda \in \mathbb{R}$). Show that f is constant and equal to 1 and, thus, verify Euler's formula.

(b) Show that, for any $\alpha, \beta \in \mathbb{R}$:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
$$\sin(\alpha + \beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta.$$

(Hint: you might want to use Euler's formula for $e^{i(\alpha+\beta)}$.)

2. Calculate the real and imaginary parts of the following expressions:

(a) $(1 + 2i)(2 - 3i)$ **(c)** $(1 + i)^3 + (1 - i)^3$ **(e)** $\left(\frac{1}{i}\right)^{2025}$ **(g)** $\frac{2i^{19} - 10i^{12}}{1+i}$

(b) $\frac{1-3i}{1+i}$ **(d)** $\frac{1}{1+i} + \frac{2}{1-i}$ **(f)** $e^{-1025\pi i}$ **(h)** $\left(1 + \sqrt{3}i\right)^{10}$

3. Calculate the modulus and an argument of the following expressions:

(a) $5 + 5i$ **(c)** $\frac{1+\sqrt{3}i}{1-i}$

(b) $(-1 + \sqrt{3}i)^{10}$ **(d)** 3^i

4. Determine all the complex solutions of the following equations:

(a) $z^5 = 1$ **(c)** $z^2 - z + 2 = 0$ **(e)** $\frac{1}{z-i} + \frac{1}{z^2-1} = 0$

(b) $z^4 = 4 + 4i$ **(d)** $z^4 - 2z^2 + i = 0$ **(f)** $|z - 1| = |z + 1|$

5. Show that, for any $x \in \mathbb{R}$, the complex number

$$z = \frac{x + i}{x - i}$$

lies on the unit circle. Show also that every point on the unit circle except for $z = 1$ can be expressed in the above form.

6. Characterize geometrically the following subsets of $\mathbb{C}$:

(a) $\Big\{z \in \mathbb{C} : \quad |z-1| = 1\Big\}$

(b) $\Big\{z \in \mathbb{C} : \quad \frac{|z-1|}{|z-i|} = 1\Big\}$

(c) $\Big\{z \in \mathbb{C} : \quad |z-1| + |z+1| = 3\Big\}$

(d) $\Big\{z \in \mathbb{C} : \quad z = 1 + 2t + 4t^2 i \quad \text{for } \ t \in \mathbb{R}\Big\}$

7. Show the following equality between sets:

$$\Big\{z \in \mathbb{C}^* : \quad z + \frac{1}{z} \in \mathbb{R}\Big\} = \Big\{z \in \mathbb{C}^* : \quad \mathrm{Im}(z) = 0 \quad \text{or} \quad |z| = 1\Big\}.$$

Solutions

1. (a) The derivative reads:

$$\begin{aligned}
\frac{\partial f(x)}{\partial x} &= \left(\frac{\partial}{\partial x} e^{-ix}\right)(\cos(x) + i\sin(x)) + e^{-ix}\frac{\partial}{\partial x}(\cos(x) + i\sin(x)) \\
&= -ie^{-ix}(\cos(x) + i\sin(x)) + e^{-ix}(-\sin(x) + i\cos(x)) \\
&= \sin(x)\cancelto{0}{\left(e^{-ix} - e^{-ix}\right)} + i\cos(x)\cancelto{0}{\left(e^{-ix} - e^{-ix}\right)} = 0
\end{aligned}$$

So $f(x)$ is constant in x. Evaluating at $x = 0$, we have

$$f(0) = e^0\Big(\cos(0) + i\sin(0)\Big) = 1,$$

so $f(x) = e^{-ix}\Big(\cos(x) + i\sin(x)\Big) \equiv 1$, which implies that

$$e^{ix} = \cos(x) + i\sin(x).$$

(b) By using Euler's formula, we can write:

$$\begin{aligned}
z = e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} &= \{\cos(\alpha) + i\sin(\alpha)\} \cdot \{\cos(\beta) + i\sin(\beta)\} \\
&= \underbrace{\{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\}}_{\mathrm{Re}(z)} + i\underbrace{\{\cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)\}}_{\mathrm{Im}(z)}
\end{aligned}$$

It follows that:

$$\begin{aligned}
\cos(\alpha + \beta) &= \mathrm{Re}\left\{e^{i(\alpha+\beta)}\right\} = \cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta) \\
\sin(\alpha + \beta) &= \mathrm{Im}\left\{e^{i(\alpha+\beta)}\right\} = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)
\end{aligned}$$

2. **(a)** $z = (1+2i)(2-3i) = 2-3i+4i+6 = 8+i$, $\text{Re}(z) = 8$ and $\text{Im}(z) = 1$

(b) $z = \frac{1-3i}{1+i} = \frac{1-3i}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i-3i-3}{2} = -1-2i$, $\text{Re}(z) = -1$ and $\text{Im}(z) = -2$

(c) Instead of developing explicitly the power (which is also possible), we can alternatively write:

$$z = (1+i)^3 + (1-i)^3 = 2^{3/2}\left\{\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^3 + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)^3\right\}$$

$$= 2^{3/2}\underbrace{\left\{e^{i3\pi/4} - e^{-i3\pi/4}\right\}}_{2\cdot\text{Re}(e^{i3\pi/4})} = 2^{3/2} 2\cos(3\pi/4) = 2^{5/2}(-2^{-1/2}) = -4$$

leading to $\text{Re}(z) = -4$ and $\text{Im}(z) = 0$.

(d) $z = \frac{1}{1+i} + \frac{2}{1-i} = \frac{1-i}{2} + \frac{2+2i}{2} = \frac{3}{2} + \frac{i}{2}$, $\text{Re}(z) = 3/2$ and $\text{Im}(z) = 1/2$

(e) $z = \left(\frac{1}{i}\right)^{2025} = -(i^{2025}) = -((i^4)^{506} \cdot i) = -(1^{506} \cdot i) = -i$, $\text{Re}(z) = 0$ and $\text{Im}(z) = -1$

(f) $z = e^{-i1025\pi} = e^{i(-1024\pi-\pi)} = e^{-i\cdot 0 \pmod{2\pi}} \cdot e^{-i\pi} = 1 \cdot (-1) = -1$, $\text{Re}(z) = -1$ and $\text{Im}(z) = 0$

(g) $z = \frac{2i^{19}-10i^{12}}{1+i} = \frac{2i^3-10}{1+i} = \frac{(-2i-10)(1-i)}{2} = \frac{-2i-10-2+10i}{2} = -6+4i$, $\text{Re}(z) = -4$ and $\text{Im}(z) = 4$

(h) It is preferable to use the exponential form:

$$z = (1+\sqrt{3}i)^{10} = 2^{10}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{10} = 2^{10}e^{i10\pi/3} = 2^{10}e^{i(2\pi+4\pi/3)}$$

$$= 2^{10}\left(\cos\left(\pi + \frac{\pi}{3}\right) + i\sin\left(\pi + \frac{\pi}{3}\right)\right) = -2^{10}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

hence $\text{Re}(z) = -2^9$ and $\text{Im}(z) = -2^9\sqrt{3}$.

3. Recall that the polar expression of a complex number $z = x+iy$ is $|z|e^{i\theta}$, where $|z| = \sqrt{x^2+y^2}$ and θ is an argument, determined up to $2\pi\mathbb{Z}$ when $z \neq 0$; the principal argument $Arg(z)$ is chosen to lie in the interval $(-\pi, +\pi]$. In view of the fact that $\frac{z}{|z|} = e^{i\theta} = \cos(\theta) + i\sin(\theta)$, θ is determined by solving the correspinding trigonometric equations.

(a) $z = 5+5i = 5(1+i) = 5\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = 5\sqrt{2}e^{i\pi/4}$ so that $|z| = 5\sqrt{2}$ and $\text{Arg}(z) = \pi/4$

(b) $z = (-1+\sqrt{3}i)^{10} = 2^{10}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)^{10} = 2^{10}\left(e^{-i\pi/3}\right)^{10} = 2^{10}e^{-i10\pi/3} = 2^{10}e^{-i4\pi+i2\pi/3} = 2^{10}e^{i2\pi/3}$ so that $|z| = 2^{10}$ and $\text{Arg}(z) = 2\pi/3$.

(c) We multiply the fraction by the conjugate of the denominator in order to make it real and identify the real and imaginary parts:

$$z = \frac{1+\sqrt{3}i}{1-i} \cdot \frac{1+i}{1+i} = \frac{1+i+\sqrt{3}i-\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2} + i\frac{1+\sqrt{3}}{2}$$

Therefore, $|z| = \sqrt{2}$ and $\mathrm{Arg}(z) = \arctan\left(\frac{1+\sqrt{3}}{1-\sqrt{3}}\right) = -\arctan(2+\sqrt{3}) = -5\pi/12$.

(d) $z = 3^i = \left(e^{\ln(3)}\right)^i = e^{i\ln(3)}$ so that $|z| = 1$ and $\mathrm{Arg}(z) = \ln(3)$.

4. We will use the fact that any number in the complex plane not equal to 0 can be expressed in polar coordinates as $z = |z|e^{i(\mathrm{Arg}(z))}$, where $\mathrm{Arg}(z) \in (-\pi, \pi]$ is unique.

(a) $z^5 = 1$ so $|z|^5 = 1 \Rightarrow |z| = 1$ and $e^{i5\mathrm{Arg}(z)} = 1 \Rightarrow 5\mathrm{Arg}(z) \in 2\pi\mathbb{Z} \Rightarrow \mathrm{Arg}(z) \in \left\{-\frac{4\pi}{5}, -\frac{2\pi}{5}, 0, \frac{2\pi}{5}, \frac{4\pi}{5}\right\}$ (we have 5 solutions).

(b) $z^4 = 4 + 4i = 4\sqrt{2}(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) = 4\sqrt{2}e^{i(\pi/4)}$ so $|z|^4 = 4\sqrt{2} \Rightarrow |z| = (32)^{\frac{1}{8}}$ and $e^{i4\mathrm{Arg}(z)} = e^{i\frac{\pi}{4}} \Rightarrow 4\mathrm{Arg}(z) - \frac{\pi}{4} \in 2\pi\mathbb{Z}$, hence $\mathrm{Arg}(z) \in \left\{\frac{\pi}{16} - \pi, \frac{\pi}{16} - \frac{\pi}{2}, \frac{\pi}{16}, \frac{\pi}{16} + \frac{\pi}{2}\right\}$ (we have 4 solutions).

(c) By completing the square in this expression, we can write:

$$\begin{aligned} z^2 - z + 2 = 0 &\iff z^2 - z + \frac{1}{4} + \frac{7}{4} = \left(z - \frac{1}{2}\right)^2 + \frac{7}{4} = 0 \\ &\iff \left(z - \frac{1}{2}\right)^2 = -\frac{7}{4} = \frac{7}{4}e^{i\pi} \\ &\iff z_k = \frac{1}{2} + \frac{\sqrt{7}}{2}e^{i(\pi/2 + k\pi)} \quad \text{with} \quad k \in \{-1, 0\} \quad \iff z_k = \frac{1 \pm i\sqrt{7}}{2}. \end{aligned}$$

Equivalently, one can directly use the quadratic formula with the definition $i = \sqrt{-1}$.

(d) By using the same method, we have:

$$\begin{aligned} z^4 - 2z^2 + i = 0 &\iff z^4 - 2z^2 + 1 - 1 + i = (z^2 - 1)^2 - 1 + i = 0 \\ &\iff (z^2 - 1)^2 = 1 - i = \sqrt{2}e^{i(-\pi/4)} \\ &\iff z_k^2 = 1 + \sqrt[4]{2}e^{i(-\pi/8 + k\pi)} \quad \text{with} \quad k \in \{0, 1\} \end{aligned}$$

Since there is no compact form for the final answer, we just write $z_1^2 = 1 + \sqrt[4]{2}e^{-i\pi/8}$ and $z_2^2 = 1 + \sqrt[4]{2}e^{i7\pi/8}$, and see that the final answer can be written as $z_n = \sqrt{|z_1|}e^{i(\mathrm{Arg}(z_1)/2 + n\pi)}$ and $z_m = \sqrt{|z_2|}e^{i(\mathrm{Arg}(z_2)/2 + m\pi)}$ with $n \in \{0, 1\}$ and $m \in \{0, 1\}$ independently.

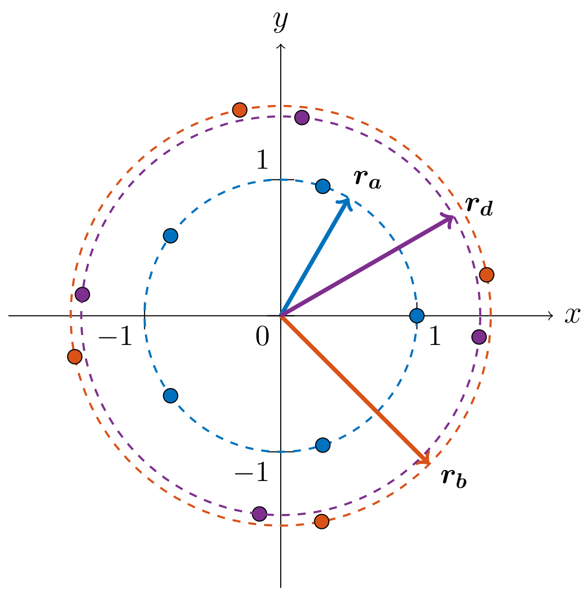
(e) Under the condition that $z \notin \{i, \pm 1\}$ (so that the fractions make sense), we write:

$$\begin{aligned} \frac{1}{z - i} + \frac{1}{z^2 - 1} = 0 &\iff \frac{z^2 + z - 1 - i}{(z - i)(z^2 - 1)} = 0 \\ &\iff z^2 + z - 1 - i = 0 \end{aligned}$$

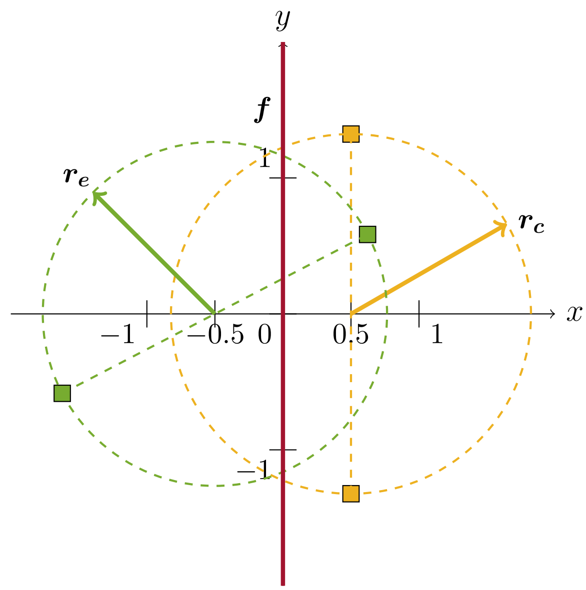
$$\iff \left(z+\frac{1}{2}\right)^2 = \frac{5}{4}+i = \frac{\sqrt{41}}{4}\left(\frac{5}{\sqrt{41}}+i\frac{4}{\sqrt{41}}\right)$$

There is no remarkable angle such that $\cos(\theta) = 5/\sqrt{41}$ and $\sin(\theta) = 4/\sqrt{41}$. We define $\theta = \arctan(4/5)$ such that the answer takes the form $z_k = -\frac{1}{2} + \frac{\sqrt[4]{41}}{2}e^{i(\theta/2+k\pi)}$ with $k \in \{0,1\}$.

(f) We write $z = x+iy$ with $(x,y) \in \mathbb{R}$ such that $\sqrt{(x+1)^2+y^2} = \sqrt{(x-1)^2+y^2}$. This expression simplifies into $x = 0$. The solutions are the purely imaginary numbers. Equivalently, one couls geometrically see the solution as the set of points in the complex plane with equal distances from $+1$ and -1; this set is the straight line passing vertically through the center of the segment connecting $+1$ and -1, which is the imaginary axis.



Solutions of the equations (a), (b) and (d)



Solutions of the equations (c), (e) and (f)

The solutions of the form $z = ae^{i\theta}$ are equally spread over a circle centered at the origin with a radius $r = a$, while those of the form $z = a + be^{i\theta}$ sit a circle of radius b shifted by a on the x axis.

5. To show that a complex number belongs to the unit circle, it is sufficient to verify if its norm is unitary:

$$|z| = \left|\frac{x+i}{x-i}\right| = \frac{|x+i|}{|x-i|} = \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} = 1\ ,\ \forall x \in \mathbb{R}$$

Any number on the unit circle not equal to 1 can be represented in this form: Let $z \neq 1$ be on the unit circle (so $|z| = 1$), then we can solve:

$$z = \frac{a+i}{a-i} \iff z(a-i) = a+i \iff a(z-1) = (z+1)i$$

$$\iff a = \frac{z+1}{z-1}i = \frac{(z+1)(\bar{z}-1)}{|z-1|^2}i = \frac{|z|^2 + z - \bar{z} - 1}{|z-1|^2}i = \frac{z-\bar{z}}{|z-1|^2}i$$

which implies that a is a *real* number (since $z - \bar{z}$ is always imaginary).

ON the other hand, $z = 1$ can not be represented in this form: Assume, for the sake of contradiction, that this is not true, then

$$\begin{aligned} 1 &= \frac{x+i}{x-i} = \frac{(x^2-1)+i2x}{x^2+1} = \frac{x^2-1}{x^2+1} + i\frac{2x}{x^2+1} \\ \implies \quad & \frac{2x}{x^2+1} = 0 \iff x = 0 \\ \implies \quad & 1 = \frac{0-1}{0+1} + i \cdot 0 = -1 \end{aligned}$$

which is obviously false. Hence, the number $z = 1$ is the only number in the unit circle that cannot be represented via this form.

6. To geometrically characterize these subsets, we write $z = x + iy$ with $(x, y) \in \mathbb{R}$ and represent the corresponding equations on the plane $\mathbb{R}^2$:

(a) $|z-1| = |x-1+iy| = \sqrt{(x-1)^2+y^2} = 1 \iff (x-1)^2 + y^2 = 1$ which is the equation of a circle of radius $r = 1$ centered in $\mathcal{C}(1, 0)$.

(c) We have:

$$\begin{aligned} \frac{|z-1|}{|z-i|} = 1 &\iff |z-1| = |z-i| \\ &\iff (x-1)^2 + y^2 = x^2 + (y-1)^2 \\ &\iff x^2 - 2x + 1 + y^2 = x^2 + y^2 - 2y + 1 \\ &\iff x = y \end{aligned}$$

which is merely the line passing by the origin and cutting the first and third quadrants in half.

(c) By squaring both sides of the equation and solving for the remaining square root, we can write:

$$\begin{aligned} |z-1| + |z+1| = 3 &\iff \sqrt{(x-1)^2+y^2} + \sqrt{(x+1)^2+y^2} = 3 \\ &\iff \sqrt{(x-1)^2+y^2} = 3 - \sqrt{(x+1)^2+y^2} \\ &\iff (x-1)^2 + y^2 = 9 - 6\sqrt{(x+1)^2+y^2} + (x+1)^2 + y^2 \\ &\iff \sqrt{(x+1)^2+y^2} = \frac{9+4x}{6} \end{aligned}$$

$$\iff (x+1)^2 + y^2 = \frac{(9+4x)^2}{36}$$

$$\iff \frac{x^2}{9/4} + \frac{y^2}{5/4} = 1$$

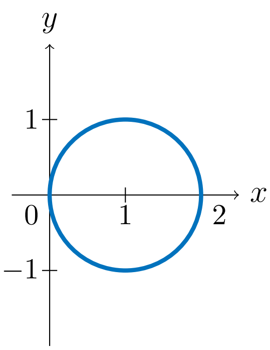
which is the standard expression for an ellipse centered at the origin with a major axis $M = 2a = 3$ and minor axis $m = 2b = \sqrt{5}$.

(d) We directly see that $\text{Re}(z) = 1 + 2t$ and $\text{Im}(z) = 4t^2$ which can be thought of as the parabolic trajectory of an object under constant positive acceleration along the y-axis. Indeed we have:

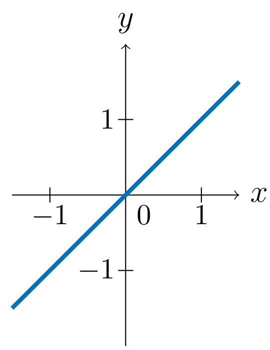
$$\vec{\mathbf{x}}(t) = \begin{pmatrix} 1+2t \\ 4t^2 \end{pmatrix}, \quad \vec{\mathbf{v}}(t) = \frac{\partial \vec{\mathbf{x}}(t)}{\partial t} = \begin{pmatrix} 2 \\ 8t \end{pmatrix}, \quad \vec{\mathbf{a}}(t) = \frac{\partial^2 \vec{\mathbf{x}}(t)}{\partial t^2} = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$$

The minimum of the parabola is at $t = 0$ and $(x, y) = (1, 0)$ for a constant positive speed along $x \sim \mathbb{R}$ and constant positive acceleration along $y \sim \mathbb{i}$.

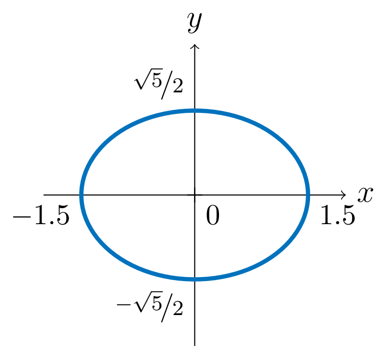
Each subset described above is graphically represented in the figures below.



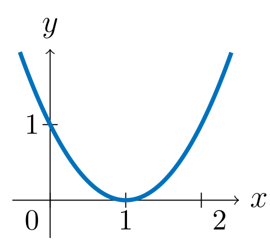
(a) $|z - 1| = 1$



(b) $|z - 1| = |z - i|$



(c) $|z - 1| + |z + 1| = 3$



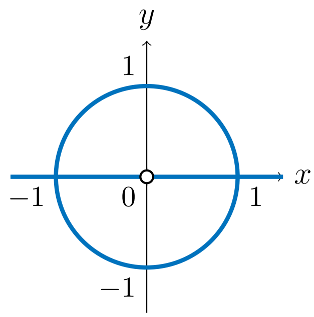
(d) $z = 1 + 2t + 4t^2 i,\ t \in \mathbb{R}$

7. As in the previous exercise, we use $z = x + iy$ and write:

$$z + \frac{1}{z} = x + iy + \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = x + iy + \frac{x-iy}{x^2+y^2}$$

$$= \frac{(x+iy)(x^2+y^2)+x-iy}{x^2+y^2}$$
$$= \frac{x^3+x(1+y^2)}{x^2+y^2} + i\frac{y(x^2+y^2-1)}{x^2+y^2} \in \mathbb{R}$$

For this complex number to be purely real, its imaginary part must be equal to zero. In doing so, we can see that there are two solutions: $y = 0$ or $x^2 + y^2 = 1$. This is equivalent to the set $\{z \in \mathbb{C}^* : \mathrm{Im}(z) = 0 \text{ or } |z| = 1\}$ that describes complex numbers that are either purely real or belong to the unitary circle centered at the origin. **Note:** the null solution $z = 0$ is discarded following the definition of the original set.



$\boldsymbol{z \in \mathbb{C}^* : z + \frac{1}{z} \in \mathbb{R}}$